Model with differential equations and convergence

Given the market model:

$$\begin{cases} D(p) = -p + 15 \\ O(p) = 1 - \dfrac{p}{3} \\ p' = 0.5[D(p) - O(p)] \end{cases}$$

Find the particular solution if $p(0) = 5$. Analyze whether $p = 5$ is a dynamically stable equilibrium, justifying your answer.

Solution

Given the market model:

$$\begin{cases} D(p) = -p + 15 \\ O(p) = 1 - \dfrac{p}{3} \\ p' = 0.5[D(p) - O(p)] \end{cases}$$

We calculate $D(p) - O(p)$:

$$\begin{aligned} D(p) - O(p) &= (-p + 15) - \left(1 - \frac{p}{3}\right) \\ &= -p + 15 - 1 + \frac{p}{3} \\ &= \left(-p + \frac{p}{3}\right) + 14 \\ &= -\frac{2p}{3} + 14 \end{aligned}$$

Then, the differential equation is:

$$p' = 0.5\left(-\frac{2p}{3} + 14\right) = -\frac{p}{3} + 7$$

Rewriting:

$$\frac{dp}{dt} + \frac{p}{3} = 7$$

$$dp = \left(7 - \frac{p}{3}\right) dt$$

$$\frac{3\,dp}{21 - p} = dt$$

$$-3\ln(21 - p) = t + C$$

$$21 - p = e^{-t/3+K}$$

$$p = 21 - e^{-t/3+K}$$

$$p = 21 + e^{-t/3}C_1$$

Applying the initial condition $p(0) = 5$:

$$5 = 21 + C_1 \implies C_1 = -16$$

Therefore, the **particular solution** is:

$$p(t) = 21 - 16e^{-\frac{t}{3}}$$

Stability Analysis:
Equilibrium is reached when $p' = 0$:

$$-\frac{p}{3} + 7 = 0 \implies p = 21$$

We observe that as $t \to \infty$:

$$e^{-\frac{t}{3}} \to 0 \implies p(t) \to 21$$

This indicates that the price converges to the equilibrium $p = 21$.